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CLASSIFICATION THEOREM FOR SMOOTH SOCIAL CHOICE*

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SUMMARY

A classification theorem for voting rules on a smooth choice space W of dimension w is presented. It is shown that, for any non-collegial voting rule, σ, there exist integers $v^*(\sigma)$, $w^*(\sigma)$ (with $v^*(\sigma) < w^*(\sigma)$) such that

(i) structurally stable σ-voting cycles may always be constructed when $w \geq v^*(\sigma) + 1$.

(ii) a structurally stable σ-core (or voting equilibrium) may be constructed when $w \leq w^*(\sigma) - 1$.

As a corollary, it is shown that a σ-voting cycle may always be constructed if W is finite and of cardinality at least $v^*(\sigma) + 2$. Finally it is shown that for an anonymous q-rule, a structurally stable core exists in dimension $\frac{n-2}{n-q}$, where n is the cardinality of the society.

CLASSIFICATION THEOREM FOR SMOOTH SOCIAL CHOICE

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INTRODUCTION

This paper proves a classification theorem for smooth social choice on a policy space, or manifold, W. By smooth social choice is meant a social preference relation, $\sigma(u)$, obtained from a smooth preference profile, u, for the society, where $\sigma(u)$ can be represented by piecewise differentiable trajectories in W. By a classification is meant an analysis of the nature of $\sigma(u)$ for all smooth profiles in terms of the dimension of the policy space. Smooth social choice includes what is often termed spatial voting games.

One method of analysis of spatial voting games has been to determine those symmetry conditions, on the direction gradients of the utility functions of the individuals at a point, sufficient to guarantee the existence of a core at the point. Results by Plott [26], Slutsky [39], Matthews [20,21] indicated, for majority rule in particular, that a core, if it did occur, would be <u>structurally unstable</u> in the sense that arbitrary small perturbations in preference would render it empty. A parallel line of enquiry considered the existence of voting cycles. A result by Kramer [15] showed, again essentially for majority rule in two dimensions, that cycles must be expected. In a series of papers by McKelvey [17,18], Cohen [3], Cohen and Matthews [4] and Matthews [22] this result was extended to show the existence of dense cycles, or voting trajectories that could wander chaotically throughout the space. Results by Schofield [30,31,34] for a general voting rule, σ, without vetoers, showed the existence of an <u>instability dimension</u> $w^*(\sigma)$, say. In particular in dimension $w^*(\sigma)$, or more, the core must be "nearly" always empty. In dimension $w^*(\sigma) + 1$ or more, dense cycles almost always occur, and in dimension $w^*(\sigma) + 2$ or more the "cycle set" is not only open dense but path connected. In these higher dimensions, a result by Matthews [22] implied that agenda manipulation could lead to almost any outcome (see also [11]).

A rather different research program has sought conditions under which a core would exist. The classical argument by Downs [8] on the existence of a core in the one-dimensional case was extended by Kramer and Klevorick [16] and Salles and Wendell [28]. In the case when the policy space was compact, convex (i.e., admissible) and preference convex then Greenberg [13] showed the existence of a <u>stability dimension</u> $v^*(\sigma)$ below which a core would exist, and above which a core need not exist, for an <u>anonymous</u> rule, σ.

In parallel to the results on voting rules for utility on policy spaces, a number of authors [6,10,25] examined the existence of a core when the alternative set, W, was of finite and restricted cardinality. In particular, Nakamura showed, for any voting rule, σ, without vetoers, that there exists an integer, the <u>Nakamura</u> number,

$v(\sigma)$, such that $|W| \leq v(\sigma) - 1$ iff the core exists for any acyclic profile on W [23].

Independently Strnad [40] and Schofield [35,36] showed, in the spatial case, that a sufficient condition for the non-emptiness of the core is that the dimension of the policy space is no greater than $v(\sigma) - 2$.

In this paper we show that the stability dimension, $v^*(\sigma)$, of a general voting rule, σ, without vetoers, is precisely $v(\sigma) - 2$. In particular, in dimension at least $v(\sigma) - 1$ it is possible to construct structurally stable cycles, in the sense that the cycles cannot be destroyed by small perturbation of preference. The method is to construct, in that dimension, a simplex which, in a certain sense, represents the voting rule. In dimension $v(\sigma) - 1$ the cycles that we consider must belong to the pareto set. Since $v(\sigma) = 3$ in general for majority rule, one of the implications is that "local" agenda manipulation cannot lead out of the pareto set, at least in two dimensions. In higher dimensions this is not necessarily the case.

The second result of the paper is to obtain a lower bound on the instability dimension. In particular we show that if σ is an anonymous q-rule (where any coalition of size q out of n players is decisive), and the dimension of the policy space is no greater than $\frac{n-2}{n-q}$ then the core may exist in a structurally stable fashion. As an immediate corollary it is evident that for such a rule the instability dimension $w^*(\sigma)$ exceeds $\frac{n-2}{n-q}$. This latter result was essentially motivated by a recent paper by Cox [5] which explored conditions

sufficient for a structurally stable core in two dimensions.

The significance of the two results lies in the classification of voting rules thus provided.

(i) In the stable dimension range, where the dimension of the policy space does not exceed $v^*(\sigma)$, then "local" cycles may not occur. For convex preferences the core will be non-empty.

(ii) In the intermediate dimension range $v(\sigma) + 1 \leq \dim(W) \leq w^*(\sigma) - 1$, structurally stable cycles exist. Moreover, even for non-convex preference, it is possible to construct a structurally stable core.

(iii) In the unstable dimension range, above $w^*(\sigma)$, a structurally stable core cannot be guaranteed.

It is true, for simple majority rule with n odd, that $v^*(\sigma) = 1$ and $w^*(\sigma) = 2$. As a consequence the intermediate dimension range is empty. However, even for majority rule, with n even, the intermediate range includes the two dimensional case. For general voting rules it is now clear that the intermediate dimension range is not unimportant.

An actual political system, which generally consists of complex "hierarchical" voting rules, will plausibly exhibit a significant intermediate dimension range. This suggests that actual political events may not, in fact, display the chaotic behavior characteristic of the unstable dimension range (see [14,41] and the articles in [24] for extensive discussion). On the contrary, political equilibrium (the core) may exist in a structurally stable fashion for some time, but this equilibrium may then "dissolve" in a "catastrophic" fashion to give way to structurally stable cycles, as preferences or political constraints slowly change. See [32] where this was earlier suggested on intuitive grounds. As long as the underlying dimension of the policy space is less than $w^*(\sigma) + 1$, then these structurally stable cycles will not be dense. Political log-rolling and agenda manipulation in this situation may well, however, bring about outcomes that are not Pareto optimal.

The next section of the paper briefly reviews recent analyses of voting rules on a finite set of alternatives, the third section presents the definitions of the key concepts and provides a statement of the classification theorem, while the fourth and fifth sections prove the results on the stability and instability dimensions.

EXISTENCE OF CYCLES AND THE CORE

Throughout the paper we shall use the definitions, notation and terminology of [31] to which the reader is referred. We review these definitions as follows.

A <u>strict preference</u> P on a set of alternatives, W, is a binary relation on W which is <u>irreflexive</u> (i.e., xPx for no x in W) and <u>asymmetric</u> (i.e., xPy implies not(yPx) for any x,y in W).
A preference, P, is <u>transitive</u> if xPy and yPz implies xPz, for any $x,y,z \in W$. A subset $\{x_1,\ldots,x_r\}$ of W is a <u>P-cycle</u> iff

$x_1Px_2 \cdots Px_rPx_1$. If P is a preference and W contains some P-cycle then P is said to be <u>cyclic</u> on W. If W contains no P-cycle, then P is said to be acyclic on W. A <u>profile</u> for a society $N = \{1,\ldots,i,\ldots,n\}$ is a list $P = \{P_1,\ldots,P_n\}$ of strict preferences, one for each member of the society. Let B(W) represent the class of strict preferences on W, and $B(W)^N$ the class of profiles on W whose components are strict preferences. Similarly let A(W) and $A(W)^N$ represent the class of acyclic strict preferences and acyclic profiles. In this paper the set of alternatives, W, will be assumed to be either (i) of finite cardinality, $|W|$, (ii) a subset of a topological vector space or (ii) a finite-dimensional smooth manifold. In the case that W is a subset of a topological vector space, we shall call W <u>admissible</u> if it is compact, convex and we shall use dim(W) to refer to the dimension of W, i.e., the dimension of the affine manifold spanned by W. If W is a smooth manifold then each point has a neighborhood locally isomorphic to an open set in $\mathbb{R}^w$, where w = dim(W) (see [29]).

A social preference function (SF) is a function, σ, which assigns to any profile, P, of strict preferences a strict preference, $\sigma(P)$, and moreover satisfies the independence axiom (see [31, Def. 2.2]. Here we shall principally examine <u>simple</u> social preference functions, or <u>voting rules</u>. A coalition, M, is <u>decisive</u> for a SF, σ, iff for any profile, P, and any alternatives $x,y \in W$

$$xP_iy \text{ for all } i \in M \Rightarrow x\sigma(P)y.$$

Let $\mathbb{D}_\sigma$ refer to the class of decisive coalitions for σ. If σ is such that $x\sigma(P)y \Rightarrow xP_iy$ for all i in some coalition M in $\mathbb{D}_\sigma$, then σ is said to be simple, and is referred to as a voting rule.

Examples of voting rules are as follows:

(i) σ is a q(w)-simple weighted majority rule iff each individual, i, is assigned a real valued integer weight $w(i) \geq 0$. A coalition, M, belongs to $\mathbb{D}_\sigma$ iff $w(M) = \sum_{i \varepsilon M} w(i) \geq q$, where the integer q is often called the quota of σ. We may refer to such a voting rule by the symbol $[q : w(1),\ldots,w(n)]$.

(ii) σ is a simple q-majority rule or q-rule iff σ is a simple weighted majority rule with $w(i) = 1$ for $i = 1,\ldots,n$, and $q > n/2$.

(iii) σ is the simple majority rule iff σ is the simple q-majority rule with $q = k + 1$ if n is odd and $n = 2k + 1$ or n is even with $n = 2k$.

An important question in social choice is whether an SF, σ, is acyclic in the sense that $\sigma(P) \varepsilon A(W)$ for all $P \varepsilon A(W)^N$ and appropriate W. If a point $x \varepsilon W$ belongs to a $\sigma(P)$-cycle then we shall say x belongs to the global cycle set $GC(\sigma,W,N,P)$. In the case that W is of finite cardinality, then $GC(\sigma,W,N,P)$ is empty only if the core (or global optima set)

$$GO(\sigma,W,N,P) = \{x \varepsilon W : \nexists y \varepsilon W \text{ st. } y\sigma(P)x\}$$

is non-empty [38]. We shall write $GC(\sigma,P)$ and $GO(\sigma,P)$ for these two

sets when there is no ambiguity.

Without imposing further restrictions on W or P, a necessary condition for $\sigma(P)$ to be acyclic is that $\mathbb{D}_\sigma$ be collegial. More formally, if $\mathbb{D} = \{A_1,\ldots,A_r\}$ is a class of coalitions then the intersection $C(\mathbb{D}) = A_1 \cap \cdots \cap A_r$ is called the collegium of $\mathbb{D}$. $\mathbb{D}$ is called collegial or non-collegial depending on whether $C(\mathbb{D})$ is non-empty or empty. If $C(\mathbb{D}_\sigma)$ is non-empty then the SF, σ, is called collegial and the members of $C(\mathbb{D}_\sigma)$ are known as vetoers; otherwise σ is called non-collegial. If W is a set of finite, but arbitrary, cardinality and σ is a non-collegial SF then it is always possible to find an acyclic profile P on W such that $\sigma(P)$ is cyclic [2].

Suppose now that σ is a non-collegial simple q-majority rule (i.e., $q < n$). Ferejohn and Grether [10] have shown that if W is of finite cardinality, $|W|$, with $|W| = r$ then $\sigma(P) \varepsilon A(W)$ for all $P \varepsilon A(W)^N$ iff $q > (\frac{r-1}{r})n$. Thus the q-rule, σ, is acyclic iff $|W| < \frac{n}{n-q}$. In other words $\sigma(P)$ is acyclic if $|W| < \frac{n}{n-q}$, and an acyclic profile P can be found on W whenever $|W| \geq \frac{n}{n-q}$ such that $\sigma(P)$ is cyclic. The first inequality may also be written $|W| \leq v(n,q) + 1$ where $v(n,q)$ is the largest integer which is strictly less than $\frac{q}{n-q}$.

In a recent paper Greenberg [13] extended this result by showing that for a q-rule, σ, if W is admissible, of dimension w, and each individual preference is continuous and convex (in a sense to be made precise below) then the core, $GO(\sigma,P)$, is non-empty iff $q > (\frac{w}{w+1})n$. Again this inequality can be written $\dim(W) \leq v(n,q)$.

Greenberg then obtained the Ferejohn-Grether result by embedding a finite set, W, of cardinality (w + 1), in the w-dimensional simplex.

By a quite independent procedure, Nakamura [23] obtained a cardinality restriction on W which was necessary and sufficient for the non-emptiness of the global optima set $GO(\sigma,P)$ in the case of a general voting rule.

Definition 1

If $\mathbb{D}$ is a family of subsets of N, then the Nakamura number $v(\mathbb{D})$ of $\mathbb{D}$ is defined as follows:

(i) if $C(\mathbb{D}) \neq \Phi$ then $v(\mathbb{D}) = \infty$.

(ii) if $C(\mathbb{D}) = \Phi$ then

$$v(\mathbb{D}) = \min\{|\mathbb{D}'| : \mathbb{D}' \subset \mathbb{D} \text{ and } C(\mathbb{D}') = \Phi\}.$$

If σ is a social preference function with decisive coalitions $\mathbb{D}_\sigma$, then the Nakamura number, $v(\sigma)$, of σ is defined to be $v(\mathbb{D}_\sigma)$.

Nakamura [23] then showed, in the case W had finite cardinality, $|W|$, that

(i) if $|W| \geq v(\sigma)$ then there exists a profile $P \in A(W)^N$ such that $GO(\sigma,P)$ is empty.

(ii) if $P \in A(W)^N$ and $|W| \leq v(\sigma) - 1$ then $GO(\sigma,P) \neq \Phi$.

It can readily be shown [35, lemma 7] that if σ is a q-rule, then $v(\sigma) = v(n,q) + 2$. Thus, as a corollary of Nakamura's result, the necessary and sufficient cardinality restriction on a finite set, W,

for a q-rule to exhibit a non-empty core is $|W| \leq v(n,q) + 1$.

Nakamura's extension of the Ferejohn-Grether result indicated that it should be possible to extend Greenberg's result to show, when W was admissible and preferences continuous and convex, that a core for the voting rule σ will exist iff $\dim(W) \leq v(\sigma) - 2$.

In two recent papers ([35], [36]) the sufficiency part of this result was obtained: that is to say for a general non-collegial voting rule, σ, on a policy manifold, W, it was shown that if $\dim(W) \leq v(\sigma) - 2$ then, in a certain sense, no cycles could occur. Moreover, if it is further assumed that W is admissible and preference is convex, then a core for the voting rule, σ, will be non-empty in this dimension [40].

One of the results of this paper is to show that this dimension constraint is necessary. In particular, if σ is a SF with Nakamura number $v(\sigma)$, and $\dim(W) \geq v(\sigma) - 1$ then we shall show that it is possible to construct a preference profile P an W such that $\sigma(P)$ exhibits cycles, and has an empty core. Moreover, the profile can be represented by smooth utility functions. Both the existence of cycles and emptiness of the core will be structurally stable in a sense to be made precise below.

As a direct corollary of this result we obtain an alternative proof of the necessity of the cardinality restraint for an alternative set of finite cardinality: that is to say we show that if $|W| \geq v(\sigma)$ then there exists an acyclic profile P such that $GC(\sigma,P) \neq \Phi$ and $GO(\sigma,P) = \Phi$. We also show that the $\sigma(P)$ cycle so constructed belongs to the pareto set.

In the case that W is a topological vector space, the assumption that $\dim(W) \geq v(\sigma) - 1$ is not sufficient to guarantee that a core cannot exist. Indeed, we shall show that whenever $\dim(W) < w^*(\sigma)$ then a structurally stable core may also exist.

SMOOTH SOCIAL CHOICE

The most general assumption that we make here is that the set of alternatives, W, is a smooth manifold (possibly with boundary) of dimension w. More restrictive assumptions are that W is a subset of a topological vector space, or that it is admissible (and thus compact, convex). A smooth preference for i is a smooth function $u_i : W \to \mathbb{R}$ that represents it's preference i.e., for all $y,x \in W$, $u_i(y) > u_i(x) \Leftrightarrow yP_ix$. A smooth profile for N is a function

$$u = (u_1,\ldots,u_n) : W \to \mathbb{R}^n$$

such that each u_i is a smooth preference for i. When $U(W)^N$ is endowed with the Whitney C^r-topology then we write it as $U_r(W)^N$ (see [12]).

Note in particular that when W is a topological vector space, with norm $|| \ ||$, then a neighborhood $N(u,\delta)$, in $U_1(W)^N$ of a profile $u \in U(W)^N$ includes any smooth profile v such that

$$||u_i(x) - v_i(x)|| < \delta \quad \text{and} \quad ||du_i(x) - dv_i(x)|| < \delta$$

for all $i \in N$ and all $x \in W$. In this case the differential, $du_i(x)$, of u_i at x may be identified with the gradient $\left(\frac{\partial u_i}{\partial x_1},\ldots,\frac{\partial u_i}{\partial x_w} \right)\Big|_x$.

A subset V of $U_r(W)^N$ is called residual if it is the intersection of open dense subsets of $U_r(W)^N$. Since $U_r(W)^N$ is a Baire space, any residual subset of $U_r(W)^N$ is also dense. A property K which can be satisfied by a utility profile is called generic in $U_r(W)^N$ iff the set $\{u \in U(W) : u \text{ satisfies } K\}$ is residual in $U_r(W)^N$. Note in particular that if K is a generic property than the set

$$\{u \in U(W)^N : u \text{ fails } K\}$$

is the complement of a dense set, and thus has empty interior in $U_r(W)^N$.

For any coalition, define the global optima set for coalition M at $u \in U(W)^N$ to be

$$GO(M,u) = \{x \in W : \nexists y \in W \text{ s.t. } u_i(y) > u_i(x)\ \forall i \in M\}.$$

The global optima set at u for the whole society, N, is also called the (global) Pareto set.

Suppose now that σ is a social preference function with decisive coalitions $\mathbb{D}_\sigma$. At the profile $u \in U(W)^N$ we may write $GO(\sigma,u)$ for the core, and note that $GO(\sigma,u) \subset \bigcap GO(M,u)$ where the intersection is over all M in $\mathbb{D}_\sigma$. If σ is a voting rule then this inclusion will be an equality. We may characterize $GO(M,u)$ by considering the critical (or infinitesimal) optima set for M. The direction gradient $du_i(x)$ of u_i at x may be regarded as a linear map $du_i(x) : T_xW \to \mathbb{R}$ where T_xW is the tangent space at x. In the case that $W \subset \mathbb{R}^w$ then T_xW may be identified with $\mathbb{R}^w$.

The <u>preference cone</u> of individual i at x is

$$H_i(x) = \{v \in T_xW : du_i(x)(v) > 0\},$$

and the <u>preference cone of coalition M</u> at x is

$$H_M(x) = \bigcap_{i \in M} H_i(x).$$

A smooth curve $c : [0,1] \to W$ from a point $x = c(0)$ in W to a point $y = c(1)$ in W is called an <u>optimizing curve</u> for M, at the profile u, iff at each point $z \in W$ along the curve the differential $dc(z)$ at z belongs to $H_M(z)$. In particular this implies that $u_i(t + \delta) > u_i(t)$ for $t \in [0,1)$, δ sufficiently small, and all $i \in M$. Write $y\rho_M x$ when there is an optimizing curve for M from x to y.

<u>Definition 2</u>

Let $u \in U(W)^N$, and let σ be an SF with decisive coalitions $\mathbb{D}_\sigma$.

(i) For coalition $M \subset N$ define the <u>critical</u> (or infinitesimal) <u>optima</u> set for M at u to be

$$IO(M,u) = \{x \in W : H_M(x) = \Phi\}.$$

(ii) Define the <u>local optima set</u> for M at u to be

$$LO(M,u) = \{x \in W : \exists \text{ nbd. } V \text{ of } x \text{ st } y\rho_M x \text{ for no } y \in V\}.$$

(iii) Define the <u>critical</u> and <u>local cores</u>, respectively, of $\sigma(u)$ by

$$IO(\sigma,u) = \bigcap IO(M,u)$$

and

$$LO(\sigma,u) = \bigcap LO(M,u),$$

where the intersections are taken over all M in $\mathbb{D}_\sigma$.

(iv) Define the <u>local cycle set</u> $LC(\sigma,u)$ of $\sigma(u)$ as follows:

$$x \in LC(\sigma,u)$$

iff for any open neighborhood V of x there is a sequence $\{x_2,\ldots,x_r\}$ of points in V such that $x\rho x_2 \rho x_3, \ldots, \rho x_r \rho x$, where we write $y\rho z$ iff for some $M \in \mathbb{D}_\sigma$ there exists an M-optimizing path from z to y, which stays within V.

(v) Define the <u>critical cycle set</u>, $IC(\sigma,u)$, of σ at u as follows:

$$x \in IC(\sigma,u)$$

iff for some subset $\mathbb{D}' \subset \mathbb{D}_\sigma$ it is the case that for each $M \in \mathbb{D}'$, there exists a (non-zero) vector $v_M \in H_M(x)$ such that $\sum_{M \in \mathbb{D}'} \lambda_M v_M = 0$, where all λ_M are strictly positive real numbers.

Some comments should be made about the relationships between these sets. First of all for coalition M, and profile u, it follows that if $y\rho_M x$ then $u_i(y) > u_i(x)$ for all $i \in M$. Moreover if $H_M(x) = \Phi$ then $y\rho_M x$ for no $y \in W$. Thus for σ a social preference function

$$GO(\sigma,u) \subset LO(\sigma,u) \subset IO(\sigma,u).$$

Under certain convexity assumptions (such as strict pseudo-concavity [15]) on the utility functions it will be the case that, for each $M \subset N$,

$$GO(M,u) = IO(M,u).$$

Moreover if σ is a voting rule, and preference satisfies the appropriate convexity assumption then the global and critical optima sets for $\sigma(u)$ will coincide.

Secondly, we may characterize $IO(\sigma,u)$ and $IC(\sigma,u)$ in a somewhat different fashion. For a profile $u \in U(W)^N$, we may represent the differential $du_i(x)$ of u_i at x by a vector $p_i(x)$ in $\mathbb{R}^W$ of unit length, where $p_i(x)$ is the normal (of unit length) to the indifference surface $u_i^{-1}[u_i(x)]$ pointing in the direction of increasing u_i. For any $M \subset N$, we shall say the vectors $\{p_i(x) : i \in M\}$ are <u>semi-positively dependent</u>, iff there exists a solution to the equation

$$0 = \sum_{i\in M} \lambda_i p_i(x)$$

where all $\lambda_i \geq 0$ but not all $\lambda_i = 0$. Let $p_M(x)$ be the convex hull of $\{p_i(x) : i \in M\}$. Semi-positive dependence is equivalent to $0 \in p_M(x)$. By standard results this implies $H_M(x) = \Phi$ [31]. Let Int W stand for the interior of W. Then, for $x \in \text{Int } W$, it is the case that

$$x \in IO(M,u) \text{ iff } 0 \in p_M(x).$$

Now define the <u>directional core</u>, $p_\sigma(x)$, of $\sigma(u)$ at x to be $p_\sigma(x) = \bigcap p_M(x)$ where the intersection is over all M in $\mathbb{D}(x)$. Here

$\mathbb{D}(x) = \{M \in \mathbb{D}_\sigma : H_M(x) \neq \Phi\}$. From [30], it is known that

(i) $x \in IC(\sigma,u)$ iff $p_\sigma(x) = \Phi$.

(ii) $IC(\sigma,u)$ is open and contained in $LC(\sigma,u) \cap \text{Int } W$.

Moreover from [37], $LC(\sigma,u)$ is contained in the closure, clos $IC(\sigma,u)$, of $IC(\sigma,u)$. Since, by definition $LC(\sigma,u)$ is contained in the global cycle set, $GC(\sigma,u)$, of $\sigma(u)$, the non-emptiness of $IC(\sigma,u)$ guarantees the existence of $\sigma(u)$ cycles. On the other hand, emptiness of $IC(\sigma,u)$ guarantees the non-existence of local cycles, though not of global cycles.

In parallel fashion the emptiness of $IO(\sigma,u)$ implies the emptiness of the core $GO(\sigma,u)$. Just as in the discrete case with a finite W the non-existence of $\sigma(P)$-cycles implies the existence of a core, so in the case when W is admissible and preference is smooth does the emptiness of $IC(\sigma,u)$ imply the non-emptiness of $IO(\sigma,u)$. Previous results on the behavior of $IC(\sigma,u)$ and $IO(\sigma,u)$ for a non-collegial voting rule, σ, are summarized in the following:

<u>Summary of Previous Results</u>

Let σ be a non-collegial voting rule with Nakamura number $v(\sigma)$, and let u stand for a smooth profile (in $U(W)^N$) on a smooth manifold W of dimension w.

S1. If $w \leq v(\sigma) - 2$ then $IC(\sigma,u) = \Phi$ [35].

S2. If W is admissible and $IC(\sigma,u) = \Phi$ then $IO(\sigma,u) \neq \Phi$ [37].

S3. If W is admissible with $w \leq v(\sigma) - 2$ and P is a "convex" and "continuous" profile then $GO(\sigma,P) \neq \Phi$ [36, 40].

S4. If $w = v(\sigma) - 1$ then $IC(\sigma,u)$, if non-empty, belongs to the interior of the critical pareto set $IO(N,u)$ [35].

S5. There exists an integer $w(\sigma) \leq n - 1$ such that

(i) $w \geq w(\sigma)$ implies $\{u \in U(W)^N : IO(\sigma,u) \cap \text{Int } W = \Phi\}$ is residual in $U_r(W)^N$, for any $r \geq 1$.

(ii) $w \geq w(\sigma) + 1$ implies $\{u \in U(W)^N : IO(\sigma,u) = \Phi\}$ and $\{u \in U(W)^N : IC(\sigma,u)$ is open dense in $W\}$ are both residual in $U_r(W)^N$, for any $r \geq 1$ [31].

S6. If σ is simple q-majority rule then the integer $w(\sigma) = q$ [31]. If σ is simple majority rule with n odd then $w(\sigma) = 2$ whereas if n is even then $w(\sigma) = 3$ [34].

In S3, "continuity" of a profile, P, means that for each P_i, $i \in N$, it is the case that for any $x \in W$, the set $\{y \in W : xP_iy\}$ is open in W. "Convexity" of P means that for each $i \in N$ and each $x \in W$ there exists a vector $v_i(x) \in W$ such that the set $\{y \in W : v_i(x) \cdot (y - x) > 0\}$ contains $\{y \in W : yP_ix\}$. S3 follows from S1 and S2 together with the convexity assumption. This result was independently obtained by Strnad [40] and generalizes the result of Greenberg mentioned in the previous section.

Our purpose in this paper is to extend these results by showing that any voting rule can be classified by two integers, the stability, and instability integers, $v^*(\sigma)$ and $w^*(\sigma)$ respectively.

Definition 3

Let σ be a voting rule.

(i) The stable subspace of profiles on W is defined to be

$$O(\sigma,W) = \{u \in U(W)^N : IO(\sigma,u) \neq \Phi\}.$$

If u belongs to Int $O(\sigma,W)$, the interior of $O(\sigma,W)$, in $U_1(W)^N$, then σ is said to have a structurally stable optima set at the profile, u.

(ii) The unstable subspace of profiles on W is defined to be

$$C(\sigma,W) = \{u \in U(W)^N : IC(\sigma,u) \neq \Phi\}.$$

If u belongs to Int $C(\sigma,W)$, the interior of $C(\sigma,W)$, in $U_1(W)^N$, then σ is said to have a structurally stable cycle set at u.

Classification Theorem for Voting Rules

Let σ be any non-collegial voting rule. Then there exist finite integers $v^*(\sigma)$ and $w^*(\sigma)$, where $v^*(\sigma) < w^*(\sigma)$, called the stability and instability integers, respectively, such that,

(i) Int $O(\sigma,W) \neq \Phi$ for every smooth manifold, W, of dimension w if and only if $w \leq w^*(\sigma) - 1$.

(ii) Int $C(\sigma,W) \neq \Phi$ for every smooth manifold, W, of dimension w if and only if $w \geq v^*(\sigma) + 1$.

The necessity aspect of (i) requires that if $w \geq w^*(\sigma)$ then there exists a manifold W, of dimension w, such that Int $O(\sigma,W) = \Phi$. This essentially is known from previous results. For sufficiency in (i) we need to show that for any smooth manifold W, if $\dim(W) \leq w^*(\sigma) - 1$ there there exists a structurally stable optima set.

For necessity in (ii) it must be shown that if $w \leq v^*(\sigma)$ then Int $IC(\sigma,W) = \Phi$ for any smooth w-dimensional manifold. If $w \leq v(\sigma) - 2$ then $IC(\sigma,W) = \Phi$, and so the candidate for $v^*(\sigma)$ is $v(\sigma) - 2$. We shall show this is the case, by demonstrating that $\dim(W) \geq v^*(\sigma)$ implies that a structurally stable cycle set exists in W. Suppose now that W is a smooth manifold, without boundary, of the dimension $w(\sigma)$ given in S5. By that result $\{u \in U(W)^N : IO(\sigma,u) = \Phi\}$ is residual and thus dense in $U_1(W)^N$. Thus $O(\sigma,W)$ may contain no interior. Consequently $w^*(\sigma) \leq w(\sigma)$. Note also that (S2) does not imply that for any manifold W of dimension $w \leq v(\sigma) - 2$ it is the case that a smooth profile u can be found in Int $IO(\sigma,u)$. However, it is reasonable to expect that $v^*(\sigma) \leq w^*(\sigma) - 1$. We show this to be the case. In particular for a non-collegial voting rule, σ, let $q = \min\{|M| : M \in \mathbb{D}_\sigma\}$, and define $s(\sigma)$ to be $\frac{n-2}{n-q}$. We shall show that $s(\sigma) < w^*(\sigma)$. In other words if W is a manifold with dimension $w \leq s(\sigma)$ then there exists a smooth profile in Int $O(\sigma,W)$. More specifically we shall prove the following theorem in the next two sections.

<u>Theorem 1</u>: Suppose that σ is a non-collegial voting rule with Nakamura number $v(\sigma)$. (i) If W is any smooth manifold of dimension $w \geq v(\sigma) - 1$ then there exists a smooth profile in Int $C(\sigma,W)$. (ii) If W is a smooth manifold of dimension $w \leq s(\sigma)$ then there exists a smooth profile in Int $O(\sigma,W)$.

Theorem 1, together with previous results, implies the classification theorem. To illustrate Theorem 1 consider the case of simple majority rule. If n is odd $(n = 2k + 1)$ then $q = k + 1$ and $s(\sigma) = \frac{2k-1}{k} = 2 - 1/k$. Thus only in 1 dimension may a structurally stable optima set be found. If n is even $(= 2k)$ then $q = k + 1$ and $s(\sigma) = \frac{2k-2}{k-1} = 2$, and so a structurally stable optima set may be found in two dimensions but not in three dimensions. Now consider a q-rule, σ. For $q = n - 1$ we obtain $s(\sigma) = n - 2$, and thus $w^*(\sigma) = w(\sigma) = n - 1$. For $q \leq n - 2$, clearly $s(\sigma) = \frac{n-2}{n-q} \geq \frac{q}{n-q} > v^*(\sigma)$, and so a structurally stable optima set may, in some cases, be found in dimension greater than $v^*(\sigma)$.

Although theorem 1(ii) provides a lower bound on $w^*(\sigma)$, there is no known procedure to compute $w^*(\sigma)$, as yet, for a general non-collegial voting rule. There is some indication, however, at least for a q-rule, that $w^*(\sigma) = 2q - n + 1$.

STRUCTURALLY STABLE CYCLES

In this section we shall show that in dimension $v(\sigma) - 1$ there exists a smooth profile u, with a structurally stable cycle set, under the voting rule σ. We shall also deduce a number of corollaries of this result and discuss its significance.

The important construction that we shall use is that of the σ-complex.

We shall show that whenever $\mathbb{D}$ is a family of subsets of N with Nakamura number $v(\mathbb{D})$ then there is a complex in dimension $v(\sigma) - 1$ which represents $\mathbb{D}$.

First of all the abstract simplex, $\Delta(Y)$, of dimension $v - 1$ in $\mathbb{R}^W$, is the convex hull of a set Y of v distinct vertices $\{y_1,\ldots,y_v\}$. Say $\Delta(y)$ is spanned by y. Opposite the vertex y_j is the face $F(j) = \Delta(Y\backslash\{y_j\})$ of dimension $v - 2$. We shall use the term edge for an intersection of faces; thus if R is a subset of $V = \{1,\ldots,v\}$ then the edge

$$F(R) = \bigcap_{j\varepsilon R} F(j)$$

is a simplex itself of dimension $v - 1 - |R|$ opposite $\{y_j : j \varepsilon R\}$. In particular if $R = \{1,\ldots,j-1, j+1,\ldots,v\}$ then $F(R) = \{y_j\}$ and if $|R| = v$ then $F(R) = \Phi$. If $\Delta(Y')$ is a simplex, spanned by $Y' \subset Y$, then the barycenter of $\Delta(Y')$ is the point

$$\theta(\Delta(Y')) = \frac{1}{|Y|} \sum_{y_j \varepsilon Y'} y_j.$$

A complex, Δ, of dimension $v - 1$, based on $Y = \{y_1,\ldots,y_j,\ldots,y_s\}$ is a family Δ of abstract simplices $\{\Delta(Y_r) : Y_r \subset Y\}$ where each simplex $\Delta(Y_r)$ in Δ has dimension at most $v - 1$, and Δ is closed under intersection. That is to say, if $\Delta(Y_j)$, $\Delta(Y_k) \varepsilon \Delta$ then $\Delta(Y_j) \cap \Delta(Y_k) \varepsilon \Delta$.

If $\mathbb{D}$ is a family of subsets (or decisive coalitions) of N, let $\mathbb{D}_m$ be the minimal subfamily of $\mathbb{D}$ (known also as the set of minimal decisive coalitions). Thus $M \varepsilon \mathbb{D}_m$ iff $M \varepsilon \mathbb{D}$ and for no $V \subset N$ is $M\backslash V \varepsilon \mathbb{D}$. We shall also refer to any subfamily $\mathbb{D}'$ of $\mathbb{D}_m$ as a minimal subfamily. We now show that we may represent $\mathbb{D}$ by a complex $\Delta(\mathbb{D})$.

Definition 4

Let $\mathbb{D}$ be a family of subsets of N, with Nakamura number, v. A representation of $\mathbb{D}$ is a complex Δ of dimension $(v - 2)$ in $\mathbb{R}^{v-1}$, spanned by $Y = \{y_1,\ldots,y_s\}$, and a surjective correspondence

$$\phi : 2^N \to \Delta \text{ such that}$$

(i) ϕ is a bijective correspondence between $\mathbb{D}_m$ and the set of faces of Δ (i.e., to each $M \varepsilon \mathbb{D}_m$ there is exactly one face (or $(v - 2)$ dimensional simplex) $\phi(M)$

(ii) for any $\mathbb{D}' \subset \mathbb{D}_m$ with $C(\mathbb{D}') \neq \Phi$ then

$\phi(C(\mathbb{D}')) = \phi(\mathbb{D}')$

where

$\phi(\mathbb{D}') = \bigcap \phi(M)$

and the intersection is over all $M \varepsilon \mathbb{D}'$.

(iii) if $\mathbb{D}_i = \{M \varepsilon \mathbb{D}_m : i \varepsilon M\} \neq \Phi$ then

$\phi(\{i\}) = \theta(\phi(\mathbb{D}_i)) = \theta(\phi(C(\mathbb{D}_i)))$.

(iv) if $\mathbb{D}_i = \Phi$ then $\Phi(\{i\})$ is the "isolated" vertex y_i in Δ.

Write $\phi : \mathbb{D} \to \Delta(\mathbb{D})$ for a representation of $\mathbb{D}$. If σ is an SF, with decisive coalitions $\mathbb{D}_\sigma$, and $\phi : \mathbb{D}_\sigma \to \Delta(\mathbb{D}_\sigma)$ represents $\mathbb{D}_\sigma$, then say $\Delta(\mathbb{D}_\sigma)$ represents σ, call Δ the σ-complex and write $\Delta(\mathbb{D}_\sigma)$ as Δ_σ.

Lemma 1

Let $\mathbb{D}$ be a family of subsets of N, with Nakamura number $v < \infty$, and let $\mathbb{D}'$ be a minimal subfamily of $\mathbb{D}$ with $|\mathbb{D}'| = v$ and $C(\mathbb{D}') = \Phi$. Then there exists a simplex $\Delta(Y)$ of dimension $(v - 1)$ spanned by $Y = \{y_1,\dots,y_v\}$ and a representation $\sigma : \mathbb{D}' \to \Delta(\mathbb{D}')$ where $\Delta(\mathbb{D}')$ is the complex based on the faces of $\Delta(Y)$. Moreover,

(i) there exists a subset $V = \{1,\dots,v\}$ of N such that for each $j \in V$, $\sigma(\{j\}) = y_j$ is a vertex of $\Delta(Y)$

(ii) after suitably labelling the sets in $\mathbb{D}'$ it is the case that for each $M_j \in \mathbb{D}'$, $\sigma(M_j) = F(j)$ the face of $\Delta(Y)$ opposite y_j

(iii) for each $i \in N$ with $\mathbb{D}_i = \{M \in \mathbb{D}' : i \in M\} \neq \Phi$

then $\quad \sigma(\{i\}) = \theta(\Delta(Y_i))$,

for the set $Y_i = \{y_j \in Y : j \in V_i\}$,

if and only if $i \in M_j$ for each $j \notin V_i$ and $i \notin M_j$ for each $j \in V_i$.

Proof. Clearly if $v(\mathbb{D}) = v$ then there exists a subfamily $\mathbb{D}'$ of $\mathbb{D}_m$ with cardinality $|\mathbb{D}'| = v$ and $C(\mathbb{D}') = \Phi$. Let $\Delta(Y)$ be the abstract simplex of dimension $(v - 1)$ spanned by $\{y_1,\dots,y_v\}$ in $\mathbb{R}^w$, where $w \geq v - 1$. Since $\Delta(Y)$ has v faces, each coalition $M_j \in \mathbb{D}'$ may be assigned by σ to exactly one face F(j) of $\Delta(Y)$. Consider any $M_j \in \mathbb{D}'$ and let $\mathbb{D}^j = \mathbb{D}'\backslash\{M_j\}$. By definition, $|\mathbb{D}^j| = v - 1$, and so $C(\mathbb{D}^j) \neq \Phi$. Moreover, a family of $(v - 1)$ faces of $\Delta(Y)$ has

intersection in a vertex of $\Delta(Y)$. We may relabel vertices such that for each $j \in V = \{1,\dots,v\}$, $y_j = \sigma(\mathbb{D}^j)$ where as before $\sigma(\mathbb{D}^j) = \bigcap_{M\in\mathbb{D}^j} \sigma(M)$. For $i \in C(\mathbb{D}^j)$ define $\sigma(\{i\}) = y_j$. From each $C(\mathbb{D}^j)$ we may choose an individual to be labelled j. By the construction, for each $j \in V = \{1,\dots,v\} \subset N$ it is the case that $\sigma(\{j\}) = y_j$, where $j \in M_k$ for $k \in V\backslash\{j\}$, and $j \notin M_j$. Clearly for each $j \in V$, and each $i \in C(\mathbb{D}^j)$, $\sigma(\{i\})$ satisfies part (iii) of the lemma. Now consider $i \in N\backslash V$ such that $\mathbb{D}_i = \{M \in \mathbb{D}' : i \in M\}$ satisfies $1 \leq |\mathbb{D}_i| \leq v - 2$. By definition $\sigma(\mathbb{D}_i) = \sigma(C(\mathbb{D}_i))$ is a simplex of dimension at least one. Suppose that $\sigma(\mathbb{D}_i) = \Delta(Y_i)$ where $Y_i = \{y_j \in Y : j \in V_i\}$. By the construction, $i \in M_j$ for each $j \notin V_i$ and $i \notin M_j$ for each $j \in V_i$. Define $\sigma(\{i\}) = \theta(\sigma(\mathbb{D}_i))$ to complete the proof of the lemma.

Q.E.D.

Theorem 2

Let σ be a non-collegial voting rule, with Nakamura number $v(\sigma)$. Then there is a σ-complex Δ_σ, of dimension $v(\sigma) - 2$ in $\mathbb{R}^w$, for $w \geq v(\sigma) - 1$, which represents σ. Moreover, for each $i \in N$, $\sigma(\{i\})$ satisfies the properties presented in lemma 1.

Proof: Let $\mathbb{D}_\sigma$ be the family of decisive coalitions of σ. For some non-collegial subfamily $\mathbb{D}' \subset \mathbb{D}_\sigma$ of minimal decisive coalitions, with $|\mathbb{D}'| = v(\sigma)$, construct the complex $\Delta(\mathbb{D}')$ representing $\mathbb{D}'$, using the procedure of lemma 1. Extend the representation $\sigma : \mathbb{D}' \to \Delta(\mathbb{D}')$ to

$\phi : \mathbb{D}_m \to \Delta(\mathbb{D}_m)$ by adding new faces and vertices as required. In particular if $i \in M$ for no $M \in \mathbb{D}_m$ define $\phi(\{i\})$ to be an isolated vertex y_i.

Q.E.D.

To illustrate the theorem consider the following example.

Example 1

Consider the voting rule with six players $\{1,2,3,4,5,6\}$ where $\mathbb{D}_\sigma = \{M_1,M_2,M_3,M_4,M_4\}$ and $M_1 = \{2,3,4\}$, $M_2 = \{1,3,4\}$, $M_3 = \{1,2,4,5\}$, $M_4 = \{1,2,3,5\}$, $M_5 = \{2,3,4,6\}$. Clearly $v(\sigma) = 4$. Choose $\mathbb{D}' = \{M_1,M_2,M_3,M_4\}$ and construct the simplex Δ spanned by $Y = \{y_1,y_2,y_3,y_4\}$ as follows.

Since $C(\mathbb{D}'\backslash\{M_j\}) = \{j\}$ for $j = 1,\ldots,4$,

we define $\phi(\{j\}) = y_j$ for $j = 1,\ldots,4$.

Thus $\phi(M_j)$ is spanned by $Y\backslash\{y_j\}$ for $j = 1,\ldots,4$. Now $\{5\} \notin M_1 \cup M_2$ and so $\phi(\{5\})$ is defined to be the barycenter of the simplex (y_1,y_2) spanned by y_1 and y_2.

Then define $\Delta(\mathbb{D}')$ to be the complex consisting of the four faces of Δ. Finally, adjoin an isolated vertex $\phi(\{6\})$ to obtain the complex Δ_σ representing σ.

As an immediate corollary of Theorem 2 we shall show that $v^*(\sigma) = v(\sigma) - 2$; that is to say we shall show that if $\dim(w) \geq v(\sigma) - 1$ then Int $C(\sigma,W) \neq \Phi$.

Corollary 1: Suppose that σ is a non-collegial social preference function with Nakamura number $v(\sigma)$. If W is a smooth manifold of dimension at least $v(\sigma) - 1$ then Int $C(\sigma,W)$ is non-empty.

Proof: Let $\mathbb{D}'$ be a minimal subfamily of $\mathbb{D}_\sigma$ with $|\mathbb{D}'| = v(\sigma)$ and $C(\mathbb{D}') = \Phi$. Furthermore, let $N' = \{i \in N : i \in M$ for some $M \in \mathbb{D}'\}$. By lemma 1, there exists a subset $V = \{1,\ldots,v\}$ of N, where $v = v(\sigma)$, and a $(v - 1)$-dimensional simplex $\Delta(Y)$ in $\mathbb{R}^{v-1}$ where $Y = \{y_i : i \in V\}$, whose faces form a complex $\Delta(\mathbb{D}')$ which represents $\mathbb{D}'$. For each $i \in N'$ define $\phi(\{i\})$ as in the proof of lemma 1. By the construction of lemma 1, it is the case that for each $M_j \in \mathbb{D}'$ and each $i \in M_j$, $\phi(\{i\}) \in \phi(M_j)$, where $\phi(M_j)$ is the face, $F(j)$, representing M_j.

Let θ be the barycenter of $\Delta(Y)$, and for each $i \in N'$ define $p_i = \phi(\{i\}) - \theta$.

For each $M_j \in \mathbb{D}'$, it is the case that the vectors $\{p_i : i \in M_j\}$ are not semi-positively dependent, and thus $0 \notin p_{M_j}$ where p_{M_j} is the convex hull of $\{p_i : i \in M_j\}$. Moreover, p_{M_j} may be identified with the face $F(j) = \phi(M_j)$.

Since $\Delta(Y)$ is in $\mathbb{R}^{v-1}$, the v distinct faces of $\Delta(Y)$ do not intersect. Thus

$$p_\sigma = \bigcap_{\mathbb{D}'} p_{M_j} = \Phi.$$

Consider a point x in the interior of the manifold, W. Since the tangent space, T_xW, is isomorphic to $\mathbb{R}^{v-1}$, we may embed the complex

$\Delta(\mathbb{D}')$ in T_xW, so that the barycenter θ coincides with the origin of T_xW. For each $i \in N'$, construct a smooth utility u_i, $W \to \mathbb{R}$ such that, with respect to an appropriate coordinate chart $du_i(x)$ is represented by a vector $p_i(x)$ in T_xW with $p_i(x) = p_i$. Assign arbitrary smooth preferences to individuals in $N \backslash N'$. By construction, $H_{M_j}(x) \neq \Phi$ for all $M_j \in \mathbb{D}'$, and so $\mathbb{D}' \subset \mathbb{D}(x)$, where as before $\mathbb{D}(x) = \{M \in \mathbb{D} : H_M(x) \neq \Phi\}$. Moreover, $p_{M_j}(x) = p_{M_j}$ for all $M_j \in \mathbb{D}'$, and so

$$p_\sigma(x) = \bigcap_{\mathbb{D}(x)} p_M(x) \subset p_\sigma = \Phi.$$

Hence

$x \in IC(\sigma,u)$ and so $x \in LC(\sigma,u)$.

Clearly the construction is stable under small perturbations of u in the C^1-topology. In other words, there exists $\delta > 0$, for δ sufficiently small, and a neighborhood $N(u,\delta)$ of u in $U_1(W)^N$ such that, for all $u' \in N(u,\delta)$, $x \in IC(\sigma,u')$. Hence Int $C(\sigma,W)$ is non-empty.

Q.E.D.

Immediately from Corollary 1 we see that $v^*(\sigma) + 1 = v(\sigma) - 1$, and so the stability dimension for a non-collegial social preference function, σ, is $v(\sigma) - 2$.

The proof of Corollary 1 and the notion of a σ-complex are essentially developed from a procedure first used by Greenberg [13] and then by Strnad [40, lemma 3] to show the existence of a smooth

profile u on an admissible set W of dimension $v(\sigma) - 1$, such that $GO(\sigma,u) = \Phi$. Their result can be slightly generalized.

Corollary 2

Suppose that σ is a non-collegial social preference function with Nakamura number $v(\sigma)$.

(i) If W is an admissible set of dimension at least $v(\sigma) - 1$, then
$C(\sigma,W)\backslash O(\sigma,W) = \{u \in U(W)^N : IC(\sigma,u) \neq \Phi \text{ and } IO(\sigma,u) = \Phi\}$
has a non-empty interior in $U_1(W)^N$.

(ii) If $\dim(W) = v(\sigma) - 1$ then there exists $u \in U(W)^N$ such that $IC(\sigma,u)$ is open dense within the global pareto set $GO(N,u)$.

Proof: In precisely the same way as Corollary 1, let $\Delta(\mathbb{D}')$ be the complex, based on $\Delta(Y)$, which represents $\mathbb{D}'$, for $\mathbb{D}'$ a non-collegial subset of $\mathbb{D}_m$, of cardinality $v = v(\sigma)$. Embed $\Delta(Y)$ in the interior of W. For each $i \in N'$, the vertex $\sigma(\{i\})$ of $\Delta(\mathbb{D}')$ may be identified with a point v_i, say, in W. Assign to individual $i \in N'$, the Type I (or Euclidean) utility function

$$u_i(x) = -||x - v_i||^2.$$

(i) To each individual $i \in N \backslash N'$ assign an arbitrary smooth preference u_i. For each $M_j \in \mathbb{D}'$, regard the face $F(j)$, which represents M_j, as a convex set in W. By definition $F(j) = \text{con}\{v_i : i \in M_j\}$. For any $x \in F(j)$, and any $i \in M_j$,

$du_i(x) = \lambda(v_i - x)$ where $\lambda > 0$. But then $\{du_i(x) : i \in M_j\}$ are semi-positively dependent. Consequently $F(j) = IO(M_j,u)$. (Indeed since the preferences of the members of M_j are convex, $F(j) = GO(M_j,u)$). Since the v faces of $\Delta(Y)$ do not intersect we obtain

$$IO(\sigma,u) \subset \bigcap_{\mathbb{D}'} IO(M_j,u) = \Phi.$$

Now consider the barycenter θ of $\Delta(Y)$. Just as in Corollary 2, for each $M_j \in \mathbb{D}'$, the convex set $p_{M_j}(\theta)$ may be identified with the face $F(j)$. Thus

$$p_\sigma(\theta) \subset \bigcap_{\mathbb{D}'} p_{M_j}(\theta) = \bigcap_{M_j \in \mathbb{D}'} F(j) = \Phi.$$

Hence

$$\theta \in IC(\sigma,u), \text{ and so } IC(\sigma,u) \neq \Phi.$$

We have now constructed a smooth profile $u \in U_1(W)^N$ such that $IO(\sigma,u) = \Phi$ and $IC(\sigma,u) \neq \Phi$. Again the construction is stable under small perturbations in the C^1-topology. Hence there exists $\delta > 0$ and a neighborhood $N(u,\delta)$ of u in $U_1(W)^N$ such that $IO(\sigma,u') = \Phi$ and $IC(\sigma,u') \neq \Phi$ for all $u' \in N(u,\delta)$. Thus the set $C(\sigma,W)\backslash O(\sigma,W)$ has non-empty interior in $U_1(W)^N$.

(ii) Suppose now that $\dim(W) = v(\sigma) - 1$. In this dimension it has already been shown [35, lemma 4] that $IC(\sigma,u) \subset IO(N,u)$ for all $u \in U(W)^N$. Proceed as in the construction of the profile u in part (i) of this proof, but to each individual $i \in N\backslash N'$ assign the Type I utility function

$$u_i(x) = -||x - v_i||^2$$

for some point v_i is the interior of $\Delta(Y)$. As a consequence $GO(N,u) = IO(N,u) = \Delta(Y)$. Moreover, for <u>any</u> point x in the interior of $\Delta(Y)$ it is the case that $p_\sigma(x) \subset \bigcap_{\mathbb{D}'} p_{M_j}(x) = \Phi$. Thus $\text{Int}\,\Delta(Y) \subset IC(\sigma,u)$. Hence $IC(\sigma,u)$ is open dense in the global pareto set $GO(N,u)$.

Q.E.D.

The second part of Corollary 2 shows, in dimension $v(\sigma) - 1$, that a profile can be constructed so that local cycles "fill" the pareto set. However, in that dimension no local cycles can occur outside the pareto set, and so iterative gradient planning procedures based on voting rules can "locate" the pareto set [see 7]. In higher dimensions, however, the cycle set need not be constrained to the pareto set and so such planning procedures may not be effective in locating the core, even if it does exist.

Note that Corollary 2(i) strengthens Corollary 1, at least in the case that W is admissible, by showing that not only has $C(\sigma,w)$ a non-empty interior, but that the complement of $O(\sigma,w)$ in $U_1(W)^N$ has a non-empty interior which intersects $C(\sigma,w)$. It is also clear that this will be the case when W is a manifold of dimension at least $v(\sigma) - 1$. Although, this result implies that $O(\sigma,w)$ cannot be open-dense in $U_1(W)^N$, it does not imply that $O(\sigma,W)$ has an empty interior in $U_1(W)^N$ in this dimension range.

Indeed in the next section of the paper we establish that, for a certain class of voting rules, $O(\sigma,W)$ has a non-empty interior in $U_1(W)^N$ in dimensions above $v(\sigma) - 1$.

First of all we use Theorem 2 to show that a core can exist in a structurally stable fashion in dimension at most $v(\sigma) - 2$, even for non-convex preference.

<u>Corollary 3</u>

Let σ be a non-collegial voting rule with Nakamura number $v(\sigma)$. If W is an admissible subset of $\mathbb{R}^w$, with $w \leq v(\sigma) - 2$, then

$$\{u \in U(W)^N : GO(\sigma,u) \neq \Phi\}$$

has a non-empty interior in $U_1(W)^N$.

<u>Proof</u>: By Theorem 2 there exists a complex Δ_σ in $\mathbb{R}^r$ for $r \geq v(\sigma) - 1$ which represents σ. Let proj: $\Delta_\sigma \to \mathbb{R}^w$ where $w \leq v(\sigma) - 2$, be a projection from $\mathbb{R}^r$ onto $\mathbb{R}^w$ such that $\text{proj}(\Delta_\sigma)$ is contained in the interior of W. Let $N' = \{i \in N : i \in M \text{ for some } M \in \mathbb{D}_\sigma\}$. To each $i \in N'$ assign the Type I smooth utility function $u_i(x) = -||x - \text{proj} \circ \mathscr{d}(\{i\})||^2$ where $\mathscr{d}(\{i\})$ is the vertex in Δ_σ representing i. To $i \in N\backslash N'$ assign an arbitrary Type I smooth utility. Thus we have constructed $u \in U(W)^N$. By the construction if $\mathbb{D}' \subset \mathbb{D}_\sigma$ with $|\mathbb{D}'| \leq v(\sigma) - 1$ then $C(\mathbb{D}') \neq \Phi$ and so $\mathscr{d}(\mathbb{D}') = \bigcap_{\mathbb{D}'} \mathscr{d}(M) \neq \Phi$. But then $\text{proj}(\mathscr{d}(\mathbb{D}')) \neq \Phi$. Since each face $\mathscr{d}(M)$ is compact, convex, so is $\text{proj}(\mathscr{d}(M))$. By Helly's Theorem [1], since $w \leq v(\sigma) - 2$,

$$\bigcap_{\mathbb{D}_\sigma} \text{proj}(\mathscr{d}(M)) \neq \Phi.$$

As in the proof of Corollary 2, if $x \in \text{proj} \circ \mathscr{d}(M)$ then the direction gradients $\{du_i(x) : i \in M\}$ are semipositively dependent. Thus $\text{proj}(\mathscr{d}(M)) = GO(M,u)$ for each $M \in \mathbb{D}_\sigma$, and so $GO(\sigma,u) \neq \Phi$. Moreover, this property is preserved under small perturbations of u in $U_1(W)^N$. Thus

$$\{u \in U(W)^N : GO(\sigma,u) \neq \Phi\}$$

has a non-empty interior in $U_1(W)^N$.

Q.E.D.

Since $GO(\sigma,u) \subset IO(\sigma,u)$, Corollary 3 establishes that σ can have a structurally stable optima set in dimension $v(\sigma) - 2$. Thus $v(\sigma) - 2 \leq w^*(\sigma) - 1$. Since $v(\sigma) - 2 = v^*(\sigma)$, by Corollary 1, we find that $v^*(\sigma) \leq w^*(\sigma) - 1$.

The final aplication of Theorem 2 is to provide a proof of Nakamura's result [23] that there exists an acyclic profile P on W such that $\sigma(P)$ is cyclic when $|W| \geq v(\sigma)$.

<u>Corollary 4</u>

Let σ be a non-collegial voting rule with Nakamura number $v(\sigma)$.

(i) If $P \in A(W)^N$ and $GC(\sigma,W,N,P) \neq \Phi$ then $|W| \geq v(\sigma)$.

(ii) If $|W| \geq v(\sigma)$ then there exists $P \in A(W)^N$ such that $GC(\sigma,W,N,P) \neq \Phi$.

(iii) If $|W| = v(\sigma)$ and $P \in A(W)^N$ such that $GC(\sigma,W,N,P) \neq \Phi$, then $GC(\sigma,W,N,P) = GO(W,N,P)$.

Proof

(i) Since $GC(\sigma,W,N,P) \neq \Phi$ there exists $\{x_1,\ldots,x_r\} \subset W$ and a $\sigma(P)$-cycle (of length r)

$$x_1\sigma(P)x_2 \cdots x_r\sigma(P)x_1.$$

Since σ is a voting rule, $x\sigma(P)y \Rightarrow xP_iy$ for all i in some coalition M_j in $\mathbb{D}_\sigma$. Write $x_r \equiv x_0$. For each $j = 1,\ldots,r$, let M_j be the decisive coalition such that $x_{j-1}P_ix_j$ for all $i \in M_j$.

Suppose that $x_i = x_j$ for some $i \neq j$. Then we obtain the cycle $x_i\sigma(P) \cdots \sigma(P)x_1\sigma(P)x_j\sigma(P) \cdots x_1$. So without loss of generality we may suppose that all $x_1,\ldots,x_r$ are distinct, all $M_1,\ldots,M_r$ are distinct and $|W| \geq r$. Let $\mathbb{D}' = \{M_1,\ldots,M_r\}$ and suppose that $C(\mathbb{D}') \neq \Phi$. Then there exists $i \in C(\mathbb{D}')$ such that

$$x_1P_ix_2 \cdots x_rP_ix_1.$$

But by assumption, $P_i \in A(W)$. By contradiction $C(\mathbb{D}') = \Phi$, and so, by definition of $v(\sigma)$, $|\mathbb{D}'| \geq v(\sigma)$. But then $r \geq v(\sigma)$ and so $|W| \geq v(\sigma)$.

(ii) Suppose that $|W| \geq v(\sigma)$. We now construct a profile $P \in A(W)^N$ and a $\sigma(P)$ cycle on W. Let $X = \{x_1,\ldots,x_v\}$, where $v = v(\sigma)$, be a subset of W of cardinality v. We adopt the

convention that $x_{v+t} \equiv x_t$, for all t. As in the proof of Corollary 1, let $\mathbb{D}'$ be a non-collegial minimal subfamily of $\mathbb{D}_\sigma$ with $|\mathbb{D}'| = v$, and construct the complex $\Delta(\mathbb{D}')$ in $\mathbb{R}^{v-1}$ which represents $\mathbb{D}'$. By definition $\mathcal{J}(\{i\}) \in \mathcal{J}(M_j)$ whenever $i \in M_j \in \mathbb{D}'$ where $\mathcal{J}(M_j) = F(j)$ is the face representing M_j. Let $Y = \{y_1,\ldots,y_v\}$ be the vertices of $\Delta(\mathbb{D}')$. For any subset Y' of Y define

$$\mathcal{J}^{-1}(Y') = \{i \in N' : \mathcal{J}(\{i\}) = \theta(\Delta(Y'))\}.$$

By theorem 2, if $i \in \mathcal{J}^{-1}(Y')$ then

$i \in M_j$ for each j s.t. $y_j \notin Y'$ and
$i \notin M_j$ for each j s.t. $y_j \in Y'$.

For each $j = 1,\ldots,v$, and to each $i \in \mathcal{J}^{-1}(y_j)$ assign the acyclic preference P_j on X which is given by

$$x_jP_jx_{j+1}P_j \cdots x_vP_jx_1 \cdots P_jx_{j-1}.$$

Consider a subset V' of $V = \{1,\ldots,v\}$ and suppose that

$$\mathcal{J}(\{i\}) = \theta(\Delta(Y')),$$

the barycenter of the simplex $\Delta(Y')$ spanned by $\{y_k \in Y : k \in V'\}$. Assign to any such individual, i, the acyclic preference $P_{V'}$ on X given by:

$xP_{V'}y$ iff xP_ky for all $k \in V'$
$xI(P_{V'})y$ iff xP_ky and yP_jx for some $j,k \in V'$.

(Here $I(P_{V'})$ is indifference). Clearly $P_{V'}$ is acyclic. Now consider a face $\sigma(M_j)$ of $\Delta(\mathbb{D}')$, spanned by $Y\backslash\{y_j\} = \{y_1,\dots,y_{j-1}, y_{j+1},\dots,y_v\}$. For each vertex $y_k (k \neq j)$ if $i \in \sigma^{-1}(y_k)$ then i is assigned a preference P_k such that $x_{j-1}P_kx_j$. Moreover, by the construction, if $i \in M_j$, then $\sigma(\{i\}) = \theta(\Delta(Y'))$ for some $Y' = \{y_k : k \in V'\} \subset Y\backslash\{y_j\}$. But for each $k \in V'$, it is the case that $x_{j-1}P_kx_j$. Thus $x_{j-1}P_{V'}x_j$. Hence $i \in M_j$ implies $x_{j-1}P_ix_j$, for $j = 2,\dots,v$. Since we have adopted the convention that $x_0 \equiv x_v$, we also find that $x_vP_ix_1$ for all $i \in M_1$. Thus for each $j = 1,\dots,v$, it is the case that $x_{j-1}P_ix_j$ for all $i \in M_j$. If $i \in M_j$ for no $M_j \in \mathbb{D}'$, assign i the preference P_i given by $xI(P_i)y$ for all $x,y \in X$. Extend the acyclic preference profile so constructed over W by defining, for all $i \in N$,

xP_iy whenever $x \in X$, $y \in W\backslash X$

$xI(P_i)y$ whenever $x,y \notin X$.

By this method we have constructed an acyclic profile for N on W such that

$$x_1\sigma(P)x_2 \cdots \sigma(P)x_v\sigma(P)x_1.$$

Clearly

$GO(\sigma,W,N,P) = \Phi$ and $GC(\sigma,W,N,P) = \{x_1,\dots,x_v\} \neq \Phi$.

(iii) Consider a $\sigma(P)$ cycle

$x_1\sigma(P)x_2 \cdots \sigma(P)x_v\sigma(P)x_1$, where $v = v(\sigma)$ as before.

Let $\mathbb{D}' = \{M_1,\dots,M_v\}$ be the associated family of decisive coalitions, such that $i \in M_j \Rightarrow x_{j-1}P_ix_j$. Clearly $C(\mathbb{D}') = \Phi$. For $j = 1,\dots,v$, consider $\mathbb{D}^j = \mathbb{D}'\backslash\{M_j\}$. Since $|\mathbb{D}^j| = v - 1$, there exists j (say) in $C(\mathbb{D}^j)$. But $j \in C(\mathbb{D}^j)$ implies

$$x_jP_jx_{j+1}P_j \cdots x_vP_jx_1 \cdots P_jx_{j-1}.$$

Thus for any $j = 1,\dots,v$ there exists an individual (labelled j) such that $x_jP_jx_{j+1}$, and an individual i, say, such that $x_{j+1}P_ix_j$. By definition $x \in GO(W,N,P)$, the pareto set in W, iff for no $y \in W$ is y pareto preferred to x (where y is <u>pareto preferred</u> to x iff yP_ix for all $i \in N$). Clearly neither x_{j+1} is pareto preferred to x_j, nor vice versa. Now choose any two alternatives x_j, $x_k = x_{j+t}$ say in W. By definition $k \in C(\mathbb{D}^k)$ has preference

$$x_k = x_{j+t}P_k \cdots P_kx_jP_k \cdots x_{k-1}.$$

Since this preference is assumed to be acyclic, it must be the case that not$(x_jP_ix_{j+t})$. Thus x_j is not pareto preferred to x_{j+t}. On the other hand $j \in C(\mathbb{D}^j)$ has preference

$$x_jP_j \cdots P_jx_{j+t}P_j \cdots x_{j-1}$$

and so not $(x_{j+t}P_jx_j)$. Hence x_{j+t} is not pareto preferred to x_j. Since $|W| = v$, $\{x_1,\ldots,x_v\} = GO(W,N,P)$. Moreover $GC(\sigma,W,N,P) = \{x_1,\ldots,x_v\}$ and so the result is proved.

Q.E.D.

Example 2

To illustrate Corollary 2, consider the voting rule, σ, of Example 1 again. As we have seen $v(\sigma) = 4$. Let W be admissible of dimension 3, and let Δ be the simplex in $\mathbb{R}^3$ whose four vertices are

$$y_1 = (-1/2,-1,-1),\ y_2 = (-1/2,1,-1),\ y_3 = (1,0,-1),\ y_4 = (0,0,1).$$

Let $y_5 = (-1/2,0,-1)$, the barycenter of y_1 and y_2, and let $y_6 = (1/6,1/3,1/3)$, the barycenter of y_2, y_3, y_4. Assign Type I preferences to the six individuals, so that $u_i(x) = -||x - y_i||^2$ for $i = 1,\ldots,6$. Clearly at the origin $\theta = (0,0,0)$ we may identify the gradient of u_i with $p_i(\theta) = y_i$. It is evident that $H_{M_j}(\theta) \neq \Phi$ for $j = 1,\ldots,5$. To show that θ belongs to the cycle set we find four vectors $v_j \in H_{M_j}(\theta)$, $j = 1,\ldots,4$ such that $\{v_1,v_2,v_3,v_4\}$ are positively dependent.

(i) $M_1 = \{2,3,4\}$. Choose $v_1 = (2,4,1)$; clearly
$p_i(\theta) \cdot v_1 > 0$ for $i \in M_1$.

(ii) $M_2 = \{1,3,4\}$. Choose $v_2 = (2,-4,1)$;
$p_i(\theta) \cdot v_2 > 0$ for $i \in M_2$.

(iii) $M_3 = \{1,2,4,5\}$. Choose $v_3 = (-4,0,1)$;
$p_i(\theta) \cdot v_3 > 0$ for $i \in M_3$.

(iv) $M_4 = \{1,2,3,5\}$. Choose $v_4 = (0,0,-1)$;
$p_i(\theta) \cdot v_4 > 0$ for $i \in M_4$.

Moreover, $v_4 + 1/3(v_1 + v_2 + v_3) = 0$. From definition 2, $\theta \in IC(\sigma,u)$. Indeed it is clear that $IC(\sigma,u)$ is the interior of $\Delta(Y)$, and $IO(\sigma,u) = \Phi$.

Example 3

To illustrate Corollary 5, consider Example 1 again, with a finite alternative set $W = \{x_1,x_2,x_3,x_4\}$. To the individuals $i = 1,2,3,4$ assign the preferences

$x_1P_1x_2P_1x_3Px_4$, $x_1P_1x_4$;
$x_2P_2x_3P_2x_4P_2x_1$, $x_2P_2x_1$;
$x_3P_3x_4P_3x_1P_3x_2$, $x_3P_3x_2$;
$x_4P_4x_1P_4x_2P_4x_3$, $x_4P_4x_3$.

To individual 5 assign the preference

$x_2P_4x_3P_5x_4I_5x_1$

and to individual 6 the preference

$x_4P_6x_1$, $x_1I_6x_2I_6x_3$

(i) $M_1 = \{2,3,4\}$: $x_4P_ix_1$ for $i \in M_1$.
(ii) $M_2 = \{1,3,4\}$: $x_1P_ix_2$ for $i \in M_2$.
(iii) $M_3 = \{1,2,4,5\}$: $x_2P_ix_3$ for $i \in M_3$.
(iv) $M_4 = \{1,2,3,5\}$: $x_3P_ix_4$ for $i \in M_4$.

Thus we obtain a cycle

$$x_1\sigma(P)x_2\sigma(P)x_3\sigma(P)x_4\sigma(P)x_1.$$

Clearly $GC(\sigma,P) = W = GO(N,P)$ and $GO(\sigma,P) = \Phi$.

THE STRUCTURALLY STABLE CORE

In this section of the paper we shall show, for a q-rule, σ, that the instability integer $w^*(\sigma)$ exceeds $\frac{n-2}{n-q}$. Let $s(n,q)$ be the greatest integer such that $s(n,q) \leq \frac{n-2}{n-q}$. We shall now show that in dimension less than or equal to $s(n,q)$ it is possible to construct a smooth profile $u \in U(W)^N$ such that σ has a structurally stable optima set at u.

<u>Theorem 3</u>

Let σ be a q-rule (with $n/2 < q \leq n - 1$) and let W be a manifold with dimension $w \leq s(n,q)$. Then Int $O(\sigma,W) \neq \Phi$.

Before proving this theorem, we require one trivial lemma and some further definitions.

<u>Lemma 2</u>

Let $t(n,q) = 2q - n + 1$. If $n/2 < q \leq n - 1$ then $s(n,q) < t(n,q)$.

<u>Proof</u>: Consider first of all the case with $(n,q) = (2k + 1,k + 1)$. As we have seen $s(n,q) < 2$. Moreover $t(n,q) = 2$ and so $s(n,q) < t(n,q)$. If $(n,q) = (2k,k + 1)$ then $s(n,q) = 2$ while

$t(n,q) = 3$. Finally consider $q = n - 1$. Clearly $s(n,q) = n - 2$, while $t(n,q) = n - 1$. Moreover, in the range $q \in (n/2,n - 1]$, $s(n,q)$ is a monotonically increasing but convex function of q, while $t(n,q)$ is linear in q. Thus $s(n,q) < t(n,q)$ for all $q \in (n/2,n - 1]$.

Q.E.D.

The procedure that we use to prove Theorem 3 depends on allocating vectors in $\mathbb{R}^w$ to members of N such that for no decisive coalition M is it the case that the vectors associated with M belong to a half space.

Suppose that $\{p_i : i \in N\}$ is a family of vectors in $\mathbb{R}^w$. As before, for any subset M of N, let
$H_M = \{v \in \mathbb{R}^w : p_i \cdot v > 0 \text{ for all } i \in M\}$, and $p_M = \text{con}\{p_i : i \in M\}$, the convex hull of the subset of vectors. We also define the <u>positive</u> and <u>negative cones</u> of p_M by

$$v \in C_+(M) \text{ iff } v = \sum_{i\in M} \lambda_i p_i \text{ where } \lambda_i > 0 \text{ for all } i \in M$$

and

$$v \in C_-(M) \text{ iff } v = \sum_{i\in M} \lambda_i p_i \text{ where } \lambda_i < 0 \text{ for all } i \in M.$$

In the obvious way say $\{p_i : i \in M\}$ <u>positively</u> span $C_+(M)$, and say they are positively dependent iff $0 \in C_+(M)$. Now let
$V = \{1,\ldots,w + 1\} \subset N$ and $\Delta(Y)$ be the w-dimensional simplex in $\mathbb{R}^w$ with barycenter, θ, at the origin of $\mathbb{R}^w$, whose vertices $Y = \{p_i : i \in V\}$ are unit vectors in $\mathbb{R}^w$. For any subset M_j of V with $|M_j| = w$, the convex hull, p_{M_j}, is the face of $\Delta(Y)$ opposite p_j.

<u>Lemma 3</u>

Let p_{M_i}, p_{M_j}, p_{M_k} be any three distinct faces of $\Delta(Y)$ in $\mathbb{R}^w$ (where $w \geq 2$). Then there exist (open) neighborhoods $U_i \subset C_-(M_i)$ and $U_k \subset C_-(M_k)$ such that, for some $v_j \in C_+(M_j)$ and any $v_i \in U_i$, $v_k \in U_k$, the vectors $\{v_i, v_j, v_k\}$ are positively dependent.

<u>Proof</u>: Since $\Delta(Y)$ is a simplex in $\mathbb{R}^w$, the set $\{p_i : i = 1,\ldots,w+1\}$ is positively dependent, and is also a positive span for $\mathbb{R}^w$.

For convenience let $M_i = V\backslash\{i\}$, for $i = 1,\ldots,w+1$. Consider any $v_j \in C_+(M_j)$. By definition v_j is positively dependent on $\{p_i : i \in V\backslash\{j\}\}$. Moreover, $\{p_i : i \in V\}$ is positively dependent, and so we can write v_j as a positive combination

$$v_j = \sum_{t=1}^{w+1} \mu_t p_t \text{ where } \mu_t > 0 \text{ for } t = 1,\ldots,w+1.$$

Now define

$$\alpha_{it} = \beta_{kt} = 1/2\ \mu_t \text{ if } t \in M_i \cap M_k.$$

$$\beta_{kt} = 0 \text{ and } \alpha_{it} = \mu_t \text{ if } t \in M_i\backslash M_k$$

$$\alpha_{it} = 0 \text{ and } \beta_{kt} = \mu_t \text{ if } t \in M_k\backslash M_i,$$

and let

$$v_i = -\sum_{t=i}^{w+1} \alpha_{it} p_t \text{ and } v_k = -\sum_{t=k}^{w+1} \beta_{it} p_t.$$

By these definitions $v_i \in C_-(M_k)$ and $v_k \in C_-(M_i)$. Moreover $v_j = -v_i - v_k$ and so $\{v_i, v_j, v_k\}$ are positively dependent. From the construction it is evident that there exist neighborhoods U_i of v_i in $C_-(M_i)$ and U_k of v_k in $C_-(M_k)$ such that $v_i' + v_k' \in C_-(M_j)$ for all $v_i' \in U_i$, $v_k' \in U_k$. But then $v_i' + v_k' + v_j' = 0$, for some $v_j' \in C_+(M_j)$ as required.

Q.E.D.

<u>Lemma 4</u>

Suppose that $w = s(n,q)$, $w \leq n-1$, $w \geq 2$ and $q \leq n-1$. Let $\mathbb{D}' = \{M \subset N\backslash\{n\} : |M| \geq q\}$, and let $\Delta(Y)$ be the simplex in $\mathbb{R}^w$ with vertices $\{p_i : i = 1,\ldots,w+1\}$ as before. Then there exist vectors $\{p_i : i = w+2,\ldots,n-1\}$ in $\mathbb{R}^w$ such that $H_M = \Phi$ for all $M \in \mathbb{D}'$.

<u>Proof</u>: Let $V = \{1,\ldots,w+1\}$ as before and let the faces p_{M_j} $(j = 1,\ldots,w+1)$ of $\Delta(Y)$ be associated with the coalitions $M_1,\ldots,M_{w+1}$. We shall say that one of these coalitions M_j, say, is <u>blocked</u> if $H_M = \Phi$ for any M containing M_j.

First of all we block the coalitions $M_1,\ldots,M_w$. To block a coalition M_j, associated with a face p_{M_j} opposite p_j, construct a <u>blocking coalition</u> $B_j \subset N\backslash V\backslash\{n\}$ of size $|B_j| = n-q-1 \geq 0$, as follows. To each member $i \in B_j$ assign a vector $p_i \in C_-(M_j)$. Clearly $\{p_k : k \in M_j\} \cup \{p_i : i \in B_j\}$ are positively dependent, and so $H_M = \Phi$ for any M containing M_j with $B_j \cap M = \Phi$. Moreover the vector p_j opposite p_{M_j} also belongs to $C_-(M_j)$. Since $|N\backslash\{n\}\backslash\{j\}\backslash B_j| = q-1$, it is clear that $H_M = \Phi$ for any $M \in \mathbb{D}$ with $M_j \subset M$. When the members of B_j are assigned vectors in $C_-(M_j)$ so as to block M_j in this way, we shall say B_j <u>blocks</u> M_j. Call $r = n-q-1$ the blocking number.

Since $w \leq \frac{n-2}{n-q}$, it is the case that $w(n - q - 1) \leq n - w - 2$. But then we may construct disjoint blocking coalitions $B_1,\ldots,B_w$ each of size $(n - q - 1)$, chosen from $N\backslash\{n\}\backslash V$, such that M_j is blocked by B_j, for $j = 1,\ldots,w$. We now block coalition M_{w+1}. Let B_{w+1} be the set of players neither in a blocking coalition nor in $N\backslash\{n\}\backslash V$. Assign each member of B_{w+1} a vector in $C\_(M_{w+1})$. Any coalition M which contains M_{w+1} together with a member of B_{w+1} must satisfy $H_M = \Phi$. Now consider M_{w+1} together with one of the blocking coalitions B_j, say. But $|M_{w+1} \cup B_j| = w + (n - q - 1) < (2q - n + 1) + (n - q - 1) = q$, since $w \leq s(n,q)$ by assumption, and $s(n,q) < 2q - n + 1$ by lemma 2. Thus if $M \in \mathbb{D}'$ and $M_{w+1} \subset M$, M must contain members of two distinct blocking coalitions.

By lemma 3, since we suppose $w \geq 2$, there exist neighborhoods U_j in $C\_(M_j)$ for $j = 1,\ldots,w$, with the following property: if $p_j \in U_j$, $p_k \in U_k$, for $j \neq k$, $j = 1,\ldots,w$, then there exists a vector $p \in C_+(M_{w+1})$ such that $\{p,p_j,p_k\}$ are positively dependent. For each blocking coalition $B_1,\ldots,B_w$ assign the members of B_j a vector in the appropriate open set U_j. By this procedure $H_M = \Phi$ for any $M \in \mathbb{D}'$ with $M_{w+1} \subset M$. Thus M_{w+1} is blocked. Now consider the family $\{B_1,\ldots,B_{w+1}\}$ of blocking coalitions. Observe that since $w = s(n,q)$ we may write

$$(n - 2) = w(n - q) + r, \text{ where } 0 \leq r \leq n - q - 1.$$

Thus

$$|B_{w+1}| = (n - w - 2) - w(n - q - 1) = r \leq n - q - 1.$$

Moreover

$$\begin{aligned}(w - 1)(n - q - 1) &= w(n - q - 1) - (n - q - 1)\\ &\leq (n - w - 2) - (n - q - 1) = q - w - 1.\end{aligned}$$

Thus no coalition consisting of the union of $(w - 1)$ of the blocking coalitions can belong to $\mathbb{D}'$. Choose the neighborhoods U_j in $C\_(M_j)$, for $j = 1,\ldots,w$ such that any family $\{p_j \in U_j : j = 1,\ldots,w\}$ is positively dependent. Thus $H_M = \Phi$ for any $M \in \mathbb{D}'$ with $\bigcup_{r=1}^{w+1} B_r \subset M$. By this construction $H_M = \Phi$ for any $M \in \mathbb{D}'$, thus proving the lemma.

Q.E.D.

<u>Proof of Theorem 3</u>

Consider the case first of all with $s(n,q) = 1$. Let W be one-dimensional. Choose a profile $u : W \to \mathbb{R}^n$ such that at x in the interior of W, $du_n(x) = 0$ and for $i = 1,\ldots,n - 1$, $du_i(x) = +1$ or -1 depending on whether i is odd or even. Clearly $H_M(x) = \Phi$ for any $|M| > n/2$. Now consider the case with $s(n,q) \geq 2$. Let $w = s(n,q)$, and let W be a smooth manifold of dimension w.

By lemma 4, it is possible to assign vectors $\{p_i : i = 1,\ldots,n - 1\}$ in $\mathbb{R}^w$ to the members of $N\backslash\{n\}$ such that $H_M = \Phi$ for any $M \in \mathbb{D}'$. Define a smooth profile $u : W \to \mathbb{R}^n$ such that, at a particular point, x, in the interior of W, $du_n(x) = 0$ and $du_i(x) = p_i$ for $i \neq n$. Clearly if $n \in M$, then $H_M(x) = \Phi$. Moreover, by lemma 4, $H_M(x) = \Phi$ for all $M \in \mathbb{D}'$. Thus $x \in IO(\sigma,u)$. Hence $u \in O(\sigma,W)$.

But then there is a neighborhood $U(u)$ of u in $U_1(W)^N$ and a neighborhood $X(x)$ of x such that for any $u' \in U(u)$, u_n' has an isolated critical point $x' \in X(x)$. The property of the simplex constructed in lemma 4, that each decisive coalition be blocked, is clearly preserved under small perturbations of the vectors. Thus for suitable chosen $U(u)$ and $X(x)$ it is the case that

$$x' \in IO(\sigma,u') \text{ for all } u' \in U(u).$$

Hence Int $IO(\sigma,w) \neq \Phi$. Finally if $w < s(n,q)$ perform the above argument in dimension $s(n,q)$ and then project down to the lower dimension.

Q.E.D.

<u>Proof of Theorem 1(ii)</u>

Let σ be a non-collegial voting rule. Let $q = \min\{|M| : M \in \mathbb{D}_\sigma\}$ and, as before, define $s(\sigma) = \frac{n-2}{n-q}$. By Theorem 3, if $\dim(W) \leq s(\sigma)$, then there exist a smooth profile u, such that, for some point $x \in W$, and any coalition M, with $|M| \geq q$, it is the case that $H_M(x) = \Phi$. Moreover, this property is structurally stable, in the sense that there exists a neighborhood $U(u)$ of u in $U_1(W)^N$ such that this property is true for any smooth profile in $U(u)$. But then $IO(\sigma,u') = \Phi$ for all $u' \in U(u)$, and so Int $O(\sigma,w) \neq \Phi$.

Q.E.D.

<u>Corollary 5</u>

Let σ be a non-collegial voting rule. If W is admissible of dimension $\leq s(\sigma)$ then $\{u \in U(W)^N : GO(\sigma,u) \neq \Phi\}$ has a non-empty interior in $U_1(W)^N$.

<u>Proof</u>: Construct the vectors $\{p_i : i = 1,\dots,n-1\}$ as in lemma 4. Choose a point x, in the interior of W, and assign individual, n, a Type I smooth utility function $u_n(y) = -||y - x||^2$. To each other individual, $i = 1,\dots,n-1$, assign a Type I utility function, such that at x,

$$du_i(x) = +\lambda_i p_i \quad \text{where} \quad \lambda_i > 0.$$

From the construction, $x \in IO(\sigma,w)$. Indeed $x \in GO(\sigma,u)$. Moreover, there will exist a neighborhood $U(u)$ of u in $U_1(W)^N$ such that $GO(\sigma,u') \neq \Phi$ for all $u' \in U(u)$.

Q.E.D.

It should also be clear that the profile, u, constructed in Corollary 5 can be chosen so that $IC(\sigma,u) = \Phi$. Thus Corollary 5 can be extended to show that

$$\{u \in U(W)^N : GO(\sigma,u) \neq \Phi \text{ and } IC(\sigma,u) = \Phi\}$$

has a non-empty interior in $U_1(W)^N$. With Corollary 2, this implies that in the dimension range $[v(\sigma) - 1, s(\sigma)]$ both $C(\sigma,w)\backslash O(\sigma,w)$ and $O(\sigma,w)\backslash C(\sigma,W)$ have non-empty interiors in $U_1(W)^N$.

Example 4

To illustrate Theorem 3, consider the q-majority rule with $(n,q) = (6,4)$. Clearly $\frac{n-2}{n-q} = 2$. We construct a structurally stable core in two dimensions. Let

$$M_1 = \{2,3\},\ M_2 = \{1,3\} \text{ and } M_3 = \{1,2\} \text{ and}$$
$$y_1 = (1,0),\ y_2 = (-\sqrt{3}/2,-1/2),\ y_3 = (-\sqrt{3}/2,1/2).$$

A blocking group for any two person coalition is of size $n - q - 1 = 1$, and so we block M_3 with $\{4\}$ and M_2 with $\{5\}$. We construct $y_4 \in C\_(M_3)$ and $y_5 \in C\_(M_2)$ such that $\{y_2,y_3,y_4,y_5\}$ are positively dependent. Let

$$y_4 = -y_2 - (\sqrt{3} - 1/2)y_1 = (1/2,1/2)$$

and

$$y_5 = -y_3 - (\sqrt{3} - 1/2)y_1 = (1/2,-1/2).$$

Clearly $\sqrt{3}(y_4 + y_5) + y_2 + y_3 = (0,0)$. Finally let $y_6 = (0,0)$. Assign Type I utility functions to the six players, so that, at a point θ in the interior of W, $du_i(\theta) = y_i$, for $i = 1,\dots,6$. Clearly $\theta \in GO(\sigma,u)$. Moreover, the core so obtained is structurally stable, and so $u \in \text{Int } O(\sigma,W)$.

Example 5

As a further illustration, consider the q rule, σ, where $q = 8$ and $n = 11$. Clearly $\frac{n-2}{n-q} = 9/3 = 3$, and we shall construct a structurally stable core in a 3 dimensional admissible policy space,

W. In a similar fashion to Example 2, let $\Delta(Y)$ be the simplex in 3-dimensions whose vertices are

$$y_1 = (-1/2,-1,-1)$$
$$y_2 = (-1/2,1,-1)$$
$$y_3 = (+1,0,-1)$$
$$y_4 = (0,0,1),$$

and embed $\Delta(Y)$ in the interior of W, so that $\Delta(Y)$ contains the origin, θ. Assign Type I utilities of the form $u_i(x) = -||x - y_i||^2$ for $i = 1,\dots,4$. As before the gradient u_i at θ is represented by $p_i(\theta) = y_i$. The blocking number of σ is $n - q - 1 = 2$, so we seek vectors y_i, $i = 5,\dots,10$ so that $\{5,6\}$, say, block $M_2 = \{1,3,4\}$, and $\{7,8\}$ block $M_1 = \{2,3,4\}$. Let
$y_5 = -1/2\ y_1 - 1/4\ y_3 - 1/4\ y_4 = (0,1/2,1/2)$ and
$y_7 = -1/2\ y_2 - 1/4\ y_3 - 1/4\ y_4 = (0,-1/2,1/2)$. Clearly $y_5 \in C\_(M_2)$ and $y_7 \in C\_(M_2)$ while $y_5 + y_7 = (0,0,1) \in C\_(1,2,3)$. Thus

$$\{y_1,y_3,y_4,y_5\},\ \{y_2,y_3,y_4,y_7\} \text{ and } \{y_1,y_2,y_3,y_5,y_7\}$$

are all sets of positively dependent vectors. Assign Type I utility functions to players $\{5,7\}$, so that $p_5(\theta) = y_5, p_7(\theta) = y_7$ and continue, in this fashion, by assigning Type I utilities to the players $i = 6,8,9,10$. Finally assign player 11 the Type 1 preference

$$u_{11}(x) = -||x||.$$

Clearly the origin θ is the core $GO(\sigma,u)$, since $H_M(\theta) = \Phi$ for all

$M \in \mathbb{D}_\sigma$.

CONCLUSION

Theorem 1 shows that, for any non-collegial voting rule, if the dimension of the policy space belongs to the range $[v(\sigma) - 1, s(\sigma)]$ then both the cycle set $IC(\sigma,u)$ and optima set $IO(\sigma,u)$ may exist in a structurally stable fashion.

For example, if we consider the case of majority rule with n even, then, as we have shown, $s(n,\sigma) = 2$, and so a structurally stable core may exist in this dimension. A recent paper by Enelow and Hinich [9] has elaborated on this observation. However, it is also known that $w^*(\sigma) = 3$ for majority rule, with n even, and so no generalization may be made from the structural stability of the core, in two dimensions, to structural stability in higher dimensions.

It is worth relating the result on the structural stability of the core in dimension at most $s(\sigma)$ to a recent theorem due to Rubinstein. Rubinstein [27] has shown that in the space of continuous profiles, with the Kannai topology, the set of profiles with non-empty core is nowhere dense. Cox [5] has recently shown how this result relates to Theorem 1(ii). Let $U_0(W)^N$ represent $U(W)^N$ with the C^0-topology. Essentially two profiles are close in $U_0(W)^N$ if their values, in $\mathbb{R}^n$, are close. The C^1-topology is finer than the C^0-topology: that is to say a set which is open in $U_0(W)^N$ will be open in $U_1(W)^N$, but there exist open sets in $U_1(W)^N$ which are not open in $U_0(W)^N$. Cox has modified Rubinstein's proof to show that the set

$\{u \in U(W)^N : GO(\sigma,u) \neq \Phi\}$ has an empty interior in $U_0(W)^N$. Note the Rubinstein-Cox result requires no dimension constraint on the policy space, W. However, as we have noted the C^1-topology on profiles is finer than the C^0-topology. In dimension no greater than $s(\sigma)$, the set of profiles

$$\{u \in U(W)^N : GO(\sigma,u) \neq \Phi\}$$

will contain a subset of profiles which is open in $U_1(W)^N$ but not open in $U_0(W)^N$. In a sense the perturbation which is required to destroy the core is "small" in the C^0-topology, but this perturbation is "large" in the C^1-topology.

Cox also obtained a restriction on q sufficient to guarantee a structurally stable core for a q-rule in two dimensions. This result provided the motivation for Theorem 1(ii).

As the reader may observe the structural stability and instability of the core and cycle set for non-collegial voting rules depends on a quite subtle fashion on the dimension of the policy space. As yet, however, an algorithm for computing the instability dimension $w^*(\sigma)$ for a general non-collegial voting rule is not available, although for simple majority rule $w^*(\sigma)$ is known precisely. Theorem 1, together with previous results, shows that $w^*(\sigma)$ must lie the the range $[s(\sigma) + 1, w(\sigma)]$. A joint paper with McKelvey [19], currently in preparation, will examine the problem of a more precise upper bound, $w(\sigma)$, for the instability integer for q-rules.

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